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A BOUNDARY ELEMENT METHOD FOR THE SOLUTION OF CONVECTIVE DIFFUSION AND BURGERS' EQUATIONS

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ABSTRACT

The paper gives the description of boundary element method (BEM) with subdomains for the solution of convection–diffusion equations with variable coefficients and Burgers' equations. At first, the whole domain is discretized into K subdomains, in which linearization of equations by representing convective velocity by the sum of constant and variable parts is carried out. Then using fundamental solutions for convection–diffusion linear equations for each subdomain the boundary integral equation (in which the part of the convective term with the constant convective velocity is not included into the pseudo-body force) is formulated. Only part of the convective term with the variable velocity, which is, as a rule, more than one order less than convective velocity constant part contribution, is left as the pseudo-source. On the one hand, this does not disturb the numerical BEM–algorithm stability and, on the other hand, this leads to significant improvement in the accuracy of solution. The global matrix, similar to the case of finite element method, has block band structure whereas its width depends only on the numeration order of nodes and subdomains. It is noted, that in comparison with the direct boundary element method the number of global matrix non-zero elements is not proportional to the square of the number of nodes, but only to the total number of nodal points. This allows us to use the BEM for the solution of problems with very fine space discretization. The proposed BEM with subdomains technique has been used for the numerical solution of one-dimensional linear steady-state convective–diffusion problem with variable coefficients and one-dimensional non-linear Burgers' equation for which exact analytical solutions are available. It made it possible to find out the BEM correctness according to both time and space. High precision of the numerical method is noted. The good point of the BEM is the high iteration convergence, which is disturbed neither by high Reynolds numbers nor by the presence of negative velocity zones.

KEY WORDS Boundary element method Convective diffusion equation Burgers' equation Boundary integral equation

NOMENCLATURE

f_i total flux;
h spatial step;
k scalar diffusivity;
L characteristic length;
P Peclet number;
q with respect to spatial co-ordinate derivative of velocity;
Re Reynolds number;
S variable convective part contribution term;
t time;
u_i velocity in direction x_i;
u temperature or concentration;
v_i convective velocity;
u^* weighting function;
x_i cartesian space co-ordinate;

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Greek symbols

Γ surrounding boundary;
δ Dirac delta function;
ε numerical error;
ξ collocation point co-ordinate;
$\Delta\tau$ time step;
τ relative time;
Ω domain enclosed by the boundary Γ.

INTRODUCTION

Boundary element methods (BEM) have spread widely during the last decade and have significant advantages over other numerical methods, such as finite difference methods (FDM) and finite element methods (FEM). However, these advantages may be realized only for the problems, described by linear differential equations. While passing from linear problems to non-linear ones for FDM and FEM the numerical scheme efficiency is impaired, but not as much as for BEM. For BEM this is connected with the fact that even if the governing differential equations system is reduced to boundary integral equation system, nevertheless it is necessary to solve the coupled system of non-linear differential equations for obtaining the weighting function.

It is clear, that for finding fundamental solutions of non-linear differential equations in the analytical form the Wiener–Hopf approach based on the consecutive usage of direct and inverse Fourier transforms to differential equations[1] would not work. The employment of alternative approaches, where fundamental solutions of non-linear equations are numerically obtained in bounded domains would not work always either. Techniques for numerical evaluation of the fundamental solutions may prove efficient enough only when this procedure is carried out only once during solution of the problem. Unfortunately, most frequently the physical variable enters into the non-linear differential equations system, describing the weighting function. In this case the numerical evaluation of the weighting function should be repeated at least several times until achieving convergence. In so doing, the computer time expenditures for finding the weight function values greatly increase and numerical methods become non-efficient.

For problems described by non-linear partial differential equations, in the boundary element methods the non-linear term was as a rule considered as a pseudo-source[1,2], whereas for the linear differential operator either the already-known fundamental solution was used or it could be found in the analytical form by the known methods. The solution for boundary and internal nodes was found in succession one after another. To evaluate the unknowns in boundary nodes it was necessary to find the inverse full matrix while the unknowns in internal nodal points could be obtained in an explicit form. In so doing, we should have used an iterative procedure for finding the solutions in all nodes, as the calculations in boundary and internal nodes were carried out separately. Nevertheless, iterations could far from always turn to be convergent. This happened if the pseudo-source term became large enough. Attempts to solve the system of equations for both boundary and internal nodes simultaneously led to a fully populated square large sized matrix, the inversion of which demanded huge computer storage.

It is necessary to note that in modern computational fluid dynamics numerical methods of finite elements are the most popular, and the computer programs created on their base are most efficient. Unfortunately, the boundary element method for solving the fluid dynamics and heat and mass transfer problems are not competitive in efficiency with the most effective finite element methods. This is explained by the fact that for solving non-linear momentum, heat and mass transfer equations, there are problems that up to the present time there are no satisfactory algorithms for boundary elements, which would allow us to avoid trouble connected with the evaluation of the convective term. It is absolutely clear that the efficiency of the calculating scheme would depend on how the non-linear convective term would be evaluated. It concerns flow particularly with high Reynolds numbers, when the convective term contribution becomes important.

The method used by Kakuda and Tosaka[3] for solving Navier–Stokes equations employing special fundamental solutions are of interest. As in the present work, non-linear equations were

linearized being discretized into subdomains, but only the constant part of the convective velocity was taken into consideration. Moreover, the question of accuracy of the numerical calculations was not investigated.

DeFigueiredo and Wrobel[4] presented the boundary element formulation for transient convective–diffusion problems using the fundamental solution of the corresponding steady-state equation and dual reciprocity approach. The paper[4] dealt only with constant velocity, while Wrobel and DeFigueiredo[5] extended the dual reciprocity boundary-element method to variable velocity fields. Similar to this, Skerget *et al.*[6] partitioned velocity field into average and variable parts; thus the convergence rate of iterations was greatly increased even at high Peclet numbers.

This paper attempts to increase the efficiency of boundary element methods for solving convective–diffusion problems with variable coefficients, which include a wide range of fluid dynamics, heat and mass transfer problems. The presented method possesses high accuracy, stability and the possibility of rather simple simulation of domains of complex configuration. Unlike BEM the global matrix has a block band structure. As this takes place, as in FEM, the width of the band depends only on the method of numbering the nodes and subdomains.

The method of solving non-linear convective–diffusion problems by dividing the whole domain Ω into K subdomains with the further linearization of transport equations in each of them and the formulation of equivalent boundary integral equations is rather simple and may be successfully used for solving Navier–Stokes and heat and mass transfer equations. This paper only considers the employment of the BEM technique with subdomains for the one-dimensional steady-state convective–diffusion problem and Burgers' equations. The investigated problems have exact analytical solutions, so numerical errors have been investigated through the paper. In the case of one-dimensional steady-state convective–diffusion equation, the numerical algorithm error depends only on the technique of convective term evaluation. That is the reason that the efficiency of employing the refinement on convective term algorithm is obvious. In the case of one-dimensional transient Burgers' equations, the total error also includes the integration with respect to time error. In this connection the efficiency of using the refinement on non-linear convective term at rather large steps with respect to time decreases and may be revealed only at very small time steps.

The boundary element method with subregions demonstrates high accuracy of numerical calculations at high Reynolds numbers, allowing the modelling of flows with shock waves without significant shock front diffusion. Besides, the numerical method does not lead to undesirable oscillations before and after the shock wave passing, which is so typical for FDM and FEM. Nevertheless, the question of employing this method to two- and three-dimensional problems is not investigated. Still it is possible to mention that for these flows the additional error, connected with spatial discretization of exact (!) boundary integral equations would not lead to abrupt lowering of the numerical accuracy. The previous experience of employing boundary elements methods gives hope for this[1,2,7].

THE STEADY-STATE CONVECTIVE DIFFUSION EQUATION

The equation of steady-state convection–diffusion in the domain $\Omega \in \mathbb{R}^n$ takes the form:

$$\frac{\partial f_i}{\partial x_i} = 0 \tag{1}$$

where n = dimension of the space $\mathbb{R}$. In (1) and elsewhere the rule of summing up with respect to repeated indexes will be employed. The total flux may be expressed as:

$$f_i = -k\frac{\partial u}{\partial x_i} + v_i u \tag{2}$$

where k = scalar diffusivity, v_i = convective velocity component, u = unknown temperature or concentration. If convective velocity and scalar diffusivity do not depend on the spatial co-ordinate x_i, then (1) transforms to a linear equation of convective diffusion with constant coefficients:

$$v_i \frac{\partial u}{\partial x_i} = k \frac{\partial^2 u}{\partial x_i \partial x_i}$$

which can be successfully solved from boundary elements methods using the corresponding fundamental solution[8,9].

And yet, in practice, cases of the convective diffusion problem with variable coefficients are more frequent. Let us consider the case when the scalar diffusivity does not depend while the convective velocity depends on the coordinate x_i. Then (1) may be rewritten in the form:

$$F(u) = \frac{\partial (P_i u)}{\partial x_i} - \frac{\partial^2 u}{\partial x_i \partial x_i} = 0 \tag{3}$$

employing the Peclet number $P_i = v_i L/k$. Then we shall consider that on the part Γ_1 of the whole border Γ, surrounding the domain Ω, values u are prescribed and on the rest part Γ_2 outer normal derivatives $\partial u/\partial n$ are prescribed.

First, we would divide all the domain Ω into K subdomains Ω_k, where $k = 1, 2, \ldots, K$. In every subdomain the variable Peclet number would be represented as the sum of constant and variable components:

$$^{(k)}P_i = {}^{(k)}\bar{P}_i + {}^{(k)}P_i' \tag{4}$$

where the Peclet number constant component is the averaged over Ω_k value:

$$^{(k)}\bar{P}_i = \frac{\int_{\Omega_k} P_i \, d\Omega}{\int_{\Omega_k} d\Omega} \tag{5}$$

Then (3) may be expressed as:

$$^{(k)}F(u) = \frac{\partial (\bar{P}_i u)}{\partial x_i} - \frac{\partial^2 u}{\partial x_i \partial x_i} + \frac{\partial (P_i' u)}{\partial x_i} \tag{6}$$

Employing the weighting residual method we obtain:

$$c(\xi)u(\xi) + \int_{\Gamma_k} u \frac{\partial u^*}{\partial n} d\Gamma + \int_{\Gamma_k} u P_i n_i u^* \, d\Gamma = \int_{\Gamma_k} \frac{\partial u}{\partial n} u^* \, d\Gamma + \int_{\Omega_k} u P_i' \frac{\partial u^*}{\partial x_i} d\Omega \tag{7}$$

In boundary integral equation (BIE) (7) the surface Γ_k surrounds the subdomain Ω_k. Note that BIE (7) is exact and suits every subdomain Ω_k, $k = 1, 2, \ldots, K$. The variable Peclet number term contribution is defined by the integral over subdomain Ω_k. The weighting function u^* is the fundamental solution of the linear convective diffusion equation:

$$^{(k)}\bar{P}_i \frac{\partial u^*}{\partial x_i} + \frac{\partial^2 u^*}{\partial x_i \partial x_i} + \delta(x, \xi) = 0 \tag{8}$$

where $\delta(x, \xi)$ is the Dirac delta function. Equation (8) solutions are well-known[9].

In this paper, we consider only one-dimensional convection–diffusion equation, for which exact analytical solutions are available. It is clear that the method described below may be without extra difficulties used both for two-dimensional and three-dimensional convection–diffusion equations.

In the one-dimensional case subdomain Ω_k represents a segment $[x_1^{(k)}, x_2^{(k)}]$, and the borders Γ_k turn into two points. BIE (7) may be rewritten then in the form:

$$c(\xi)u(\xi) + \left[uq^*\right]_{x_1^{(k)}}^{x_2^{(k)}} + \left[Puu^*\right]_{x_1^{(k)}}^{x_2^{(k)}} = \left[qu^*\right]_{x_1^{(k)}}^{x_2^{(k)}} + \int_{x_1^{(k)}}^{x_2^{(k)}} uP'q^* \, dx \tag{9}$$

where

$$q = \frac{du}{dx} \qquad q^* = \frac{du^*}{dx}$$

The fundamental solution for the one-dimensional convective–diffusion equation has the form:

$$u^*(x, \xi) = \frac{1}{|\bar{P}|} \exp\left\{-\frac{\bar{P}(x - \xi) + |\bar{P}||x - \xi|}{2}\right\} \tag{10}$$

Then we can obtain:

$$q^*(x, \xi) = -\frac{\text{sign}(\bar{P}) + \text{sign}(x - \xi)}{2} \exp\left\{-\frac{\bar{P}(x - \xi) + |\bar{P}||x - \xi|}{2}\right\} \tag{11}$$

In BIE (9) let us place the collocation point inside the subdomain Ω_k and then consider two cases of $\xi \to x_1^{(k)}$ and $\xi \to x_2^{(k)}$. In these cases we have $c(\xi) = 1$ and there will be no problems in defining the signs of difference $(x - \xi)$. Then BIE (9) may be rewritten in the final form for the kth subdomain:

$$u_m^{(k)} + u_2^{(k)} q_{m,2}^{*(k)} - u_1^{(k)} q_{m,1}^{*(k)} + P_2^{(k)} u_2^{(k)} u_{m,2}^{*(k)} - P_1^{(k)} u_1^{(k)} u_{m,1}^{*(k)} = q_2^{(k)} u_{m,2}^{*(k)} - q_1^{(k)} u_{m,1}^{*(k)} + S_m^{(k)} \tag{12}$$

where

$$S_m^{(k)} = \int_{x_1^{(k)}}^{x_2^{(k)}} P'(x)u(x)q^*(x, \xi) \, dx \tag{13}$$

$m = 1, 2$ are local numbers of segment ends $[x_1^{(k)}, x_2^{(k)}]$. It is possible to write $2K$ equations similar to (12), so there will be $4K$ unknowns. Employing $2(K - 1)$ compatibility conditions:

$$u_2^{(k)} = u_1^{(k+1)} \qquad q_2^{(k)} = q_1^{(k+1)} \qquad k = 1, 2, \ldots, K - 1 \tag{14}$$

the number of the unknowns reduces to $2K$. Then the global system would be rewritten in the form:

$$[A]\{y\} = \{b\} \tag{15}$$

where $\{y\}$ = vector of the unknowns, $\{b\}$ = vector of free coefficients. It is obtained after the substitution of known boundary conditions in BIE (12) and further term regrouping. In comparison with boundary element methods the global matrix $[A]$ being the size of $2K$ is not dense but would possess a block band structure. The matrix $[A]$ in general case is not symmetrical and thus it is necessary to store all the band width of the matrix $[A]$, which in the one-dimensional case equals five and in the more general case (two or three-dimensional problems) the band width would depend only on the way of collocation nodes and subdomains numbering.

To avoid additional nodal points, linear interpolating function for $u(x)$ is used in the paper:

$$u^{(k)}x = \theta_1(x)u_1^{(k)} + \theta_2(x)u_2^{(k)} \tag{16}$$

where

$$\theta_1(x) = \frac{x_2^{(k)} - x}{x_2^{(k)} - x_1^{(k)}}, \qquad \theta_2(x) = \frac{x - x_1^{(k)}}{x_2^{(k)} - x_1^{(k)}} \tag{17}$$

Then $S_m^{(k)}$ is brought to the form:

$$S_m^{(k)} = u_1^{(k)} \int_{x_1^{(k)}}^{x_2^{(k)}} \theta_1(x) P'(x) q^*(x, \xi)\, dx + u_2^{(k)} \int_{x_1^{(k)}}^{x_2^{(k)}} \theta_2(x) P'(x) q^*(x, \xi)\, dx \tag{18}$$

Integrals in (18) may be counted numerically using Gaussian integration as they do not possess singularities. While establishing the contribution of the variable Peclet number the nodal variables in (18) may be ordered in the left part of (15). By doing so, the system of linear algebraic equations (15) should be solved only once to get a final solution. As the matrix $[A]$ is pentadiagonal, it may be efficiently inverted both by direct (for example by Thomas algorithm) and iterative methods (for instance, by successive over-relaxation method). In the present paper, one-dimensional problems do not lead to large matrix $[A]$ and that is the reason that Gauss elimination method is used here.

BURGERS' EQUATION

The most attractive object for investigating the numerical scheme accuracy and stability, for fluid dynamics and heat transfer problems, is the Burgers' equations, representing reduced Navier–Stokes equations with zero pressure gradient:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{19}$$

The lack of pressure in (19) does not demand the attraction of additional methods for evaluating the pressure term and besides the necessity in continuity equation falls off. In spite of the fact that (19) is non-linear, exact analytical solutions may be found for it with the Cole–Hopf transformation[10] at definite boundary and initial conditions.

As in the case with steady-state convection–diffusion equations, we would linearize (19), preliminarily having divided the whole domain Ω into K subdomains and having represented the convective velocity in the form of the sum:

$$u_i^{(k)} = \bar{u}_i^{(k)} + u_i'^{(k)} \tag{20}$$

of the constant $\bar{u}_i$ and the variable u_i' parts. The constant part of the convective velocity is considered as the averaged value in the volume Ω_k:

$$\bar{u}_i^{(k)} = \frac{\int_{\Omega_k} u_i^{(k)}\, d\Omega}{\int_{\Omega_k} d\Omega} \tag{21}$$

Having used the weighted residual method with further integration with respect to time we can get BIE for Burgers' equations in the form:

$$\begin{aligned} c(\xi) u_i(\xi, 0) + \frac{1}{Re} \int_0^{\Delta\tau} \int_{\Gamma_k} u_i(x, \tau) \frac{\partial u^*(x, \xi, \tau)}{\partial n}\, d\Gamma\, d\tau + \int_0^{\Delta\tau} \int_{\Gamma_k} u_i(x, \tau) \bar{u}_j n_j u^*(x, \xi, \tau)\, d\Gamma\, d\tau \\ = \frac{1}{Re} \int_0^{\Delta\tau} \int_{\Gamma_k} \frac{\partial u_i(x, \tau)}{\partial n} u^*(x, \xi, \tau)\, d\Gamma\, d\tau + \int_{\Omega_k} u_i(x, \Delta\tau) u^*(x, \xi, \Delta\tau)\, d\Omega \\ - \frac{1}{Re} \int_0^{\Delta\tau} \int_{\Omega_k} u_j'(x, \tau) \frac{\partial u_i(x, \tau)}{\partial x_j} u^*(x, \xi, \tau)\, d\Omega\, d\tau \end{aligned} \tag{22}$$

In (22) the following time designations are used: $\tau = t_F - t$, where t_F is fixed time at which we

would like to find BIE (22) solution. Correspondingly, at time $\tau = 0$ the variables are unknown and at $\tau = \Delta\tau$ they are known.

Included in (22) the weighting function satisfies the transient convection–diffusion equation:

$$-\frac{\partial u^*}{\partial \tau} + \bar{u}_j \frac{\partial u^*}{\partial x_j} + \frac{1}{Re}\frac{\partial^2 u^*}{\partial x_j \partial x_j} + \delta(x, \xi)\delta(\tau) = 0 \tag{23}$$

Note that BIE (22) is the exact integral equation, as it includes the integral considering the variable convective velocity part contribution.

Here we would also restrict our consideration only to one-dimensional Burgers' equation, for which accurate analytical solutions are available. This consideration makes it possible to estimate the discretization error with respect to time as applied to BEM for unsteady differential equations as the discretization error with respect to the spatial co-ordinate is considered to be found from the steady-state convection–diffusion equation solution. We should note that the discretization with respect to time method is not the only one possible. The method of Laplace transformation for governing equations and also of boundary and initial conditions[2] can also be used. A further attraction is the dual reciprocity boundary element method (DRBEM), suggested by Brebbia and Wrobel[11]. In this case it is necessary to employ the weighting function, which is the fundamental solution of steady-state convection–diffusion equation.

For one-dimensional Burgers' equation the corresponding boundary integral equation may be written in the form:

$$u^{(k)}(\xi, 0) + \int_0^{\Delta\tau} \left[u\bar{u}u^* \right]_{x_1^{(k)}}^{x_2^{(k)}} d\tau + \frac{1}{Re}\int_0^{\Delta\tau} \left[uq^* \right]_{x_1^{(k)}}^{x_2^{(k)}} d\tau = \frac{1}{Re}\int_0^{\Delta\tau} \left[qu^* \right]_{x_1^{(k)}}^{x_2^{(k)}} d\tau + \int_{x_1^{(k)}}^{x_2^{(k)}} u(x, \Delta\tau)u^*(x, \xi, \Delta\tau)\, dx - \int_0^{\Delta\tau}\int_{x_1^{(k)}}^{x_2^{(k)}} qu'u^*\, dx\, d\tau \tag{24}$$

The weighting function, included in (24), and which is the fundamental solution of transient convective–diffusion equation (23), in the one-dimensional case may be written in the form:

$$u^*(x, \xi, \tau) = H(\tau)\sqrt{\frac{Re}{4\pi\tau}} \exp\left\{ -\frac{Re(x - \xi + \tau\bar{u})^2}{4\tau} \right\} \tag{25}$$

where $H(\tau)$ is Heaviside step function:

$$H(\tau) = \begin{cases} 0 & \text{at } \tau < 0 \\ 1 & \text{at } \tau \geqslant 0 \end{cases}$$

Correspondingly, the derivative of the weighting function with respect to spatial coordinate would be written in the following way:

$$q^*(x, \xi, \tau) = \frac{\partial u^*}{\partial x} = -\frac{Re(x - \xi + \tau\bar{u})}{2\tau} u^*(x, \xi, \tau) \tag{26}$$

Weighting function is $u^*(x, \xi, \tau) \geqslant 0$ at $\tau \geqslant 0$, and $u^*(x, \xi, \tau) = 0$ at $\tau < 0$.

Contrary to BIE in steady-state convective–diffusion equations, the problem of temporal interpolation from $\tau = 0$ to $\tau = \Delta\tau$ appears for the velocity $u(x, \tau)$ and its derivative $q(x, \tau) = \partial u(x, \tau)/\partial x$. Below, two ways of interpolation would be considered:

(a) variables u, q are constant in the range between 0 and $\Delta\tau$:

$$\begin{aligned} u(x, \tau) &= u(x, 0) \\ q(x, \tau) &= q(x, 0) \end{aligned} \tag{27}$$

(b) variables u, q are linearly changed from 0 and $\Delta\tau$:

$$u(x,\tau) = \varphi_1(\tau)u(x,\Delta\tau) + \varphi_2(\tau)u(x,0)$$
$$q(x,\tau) = \varphi_1(\tau)q(x,\Delta\tau) + \varphi_2(\tau)q(x,0) \tag{28}$$

where linearity with respect to time interpolating function is of the form:

$$\varphi_1(\tau) = \frac{\tau}{\Delta\tau} \qquad \varphi_2(\tau) = 1 - \frac{\tau}{\Delta\tau} \tag{29}$$

Note that in case (a) the numerical scheme has a strong implicit form. Employing the constant with respect to time interpolation the BIE (24) would be rewritten to the form:

$$u_m^{(k)} + \bar{u}^{(k)}(u_2^{(k)}?_{m,2}^{(k)} - u_1^{(k)}P_{m,1}^{(k)}) + \frac{1}{Re}(u_2^{(k)}Q_{m,2}^{(k)} - u_1^{(k)}Q_{m,1}^{(k)}) = \frac{1}{Re}(q_2^{(k)}P_{m,2}^{(k)} - q_1^{(k)}P_{m,1}^{(k)}) + V_m^{(k)} - S_m^{(k)} \tag{30}$$

where $k = 1, 2, \ldots, K$; $m = 1, 2$

$$P_{m,n}^{(k)} = \int_0^{\Delta\tau} u^*(x,\xi,\tau)\,d\tau = \frac{b_1 - b_2}{|\bar{u}|} \tag{31}$$

$$Q_{m,n}^{(k)} = \int_0^{\Delta\tau} q^*(x,\xi,\tau)\,d\tau = -\frac{Re}{4}[\{\mathrm{sign}(x-\xi) + \mathrm{sign}(\bar{u})\}b_1 + \{\mathrm{sign}(x-\xi) - \mathrm{sign}(\bar{u})\}b_2] \tag{32}$$

$$b_1 = \exp\left\{-\frac{\mathrm{Re}}{4}(r|\bar{u}| + (x-\xi)\bar{u})\right\}\left\{2 - \mathrm{erfc}\left(\frac{|\bar{u}|}{4}\sqrt{\Delta\tau Re} - \frac{r}{2}\sqrt{\frac{Re}{\Delta\tau}}\right)\right\} \tag{33}$$

$$b_2 = \exp\left\{\frac{Re}{4}(r|\bar{u}| - (x-\xi)\bar{u})\right\}\mathrm{erfc}\left(\frac{|\bar{u}|}{4}\sqrt{\Delta\tau Re} + \frac{r}{2}\sqrt{\frac{Re}{\Delta\tau}}\right) \tag{34}$$

In (33), (34) the following notations are used:

$$r = |x - \xi|$$

$\mathrm{erfc}(\cdot)$—error functions

The integral over the subdomain:

$$V_m^{(k)} = \int_{x_1^{(k)}}^{x_2^{(k)}} u(x,\Delta\tau)u^*(x,\xi,\Delta\tau)\,dx \tag{35}$$

should be evaluated at each time step as it contains the weighting function u^*, adjusted with each iteration, whereas the velocity values are already known at $\tau = \Delta\tau$. Obviously the integral (35) must be evaluated accurately enough, not to accumulate large errors. Though only the variables at the ends of the subdomains are included in the calculations, nevertheless in order to get a more accurate evaluation of (35) it is necessary to find the velocities $u(x,\tau)$ in Gauss points at time $\tau = \Delta\tau$, while using numerical Gauss quadrature for calculating $V_m^{(k)}$.

The double integral calculations:

$$S_m^{(k)} = \int_0^{\Delta\tau}\int_{x_1^{(k)}}^{x_2^{(k)}} q(x,\tau)u'(x,\tau)u^*(x,\xi,\tau)\,dx\,d\tau \tag{36}$$

considering the convective velocity variable component contribution should be repeated at each iteration. Employing the linear interpolation (16) for velocity over the subdomain and the

condition (27) at that, we would get:

$$q(x,\tau)=\frac{u_2^{(k)}-u_1^{(k)}}{x_2^{(k)}-x_1^{(k)}}$$

$$u'^{(k)}(x,\tau)=\theta_1(x)u_1'^{(k)}+\theta_2(x)u_2'^{(k)}$$

Then the integral (36) may be calculated at numerical Gauss quadrature double usage. In this paper we have also used the quadratic interpolation for $u'(x,\tau)$, $q(x,\tau)$ with the additional node in the middle of subdomain, but no considerable accuracy improvement has been achieved.

The usage of linear temporal interpolation functions allows us to rewrite BIE (24) in the form:

$$u_m^{(k)}+\bar{u}^{(k)}(u_2^{(k)}P_{2,m,2}^{(k)}-u_1^{(k)}P_{2,m,1}^{(k)})+\frac{1}{Re}(u_2^{(k)}Q_{2,m,2}^{(k)}-u_1^{(k)}Q_{2,m,1}^{(k)})$$

$$=\frac{1}{Re}(q_2^{(k)}P_{2,m,2}^{(k)}-q_1^{(k)}P_{2,m,1}^{(k)})+V_m^{(k)}-S_m^{(k)}+\bar{u}^{(k)}(u_1^{(k)}(\Delta\tau)P_{1,m,1}^{(k)}-u_2^{(k)}(\Delta\tau)P_{1,m,2}^{(k)})$$

$$+\frac{1}{Re}(u_1^{(k)}(\Delta\tau)Q_{1,m,1}^{(k)}-u_2^{(k)}(\Delta\tau)Q_{1,m,2}^{(k)})+\frac{1}{Re}(q_2^{(k)}(\Delta\tau)P_{1,m,2}^{(k)}-q_1^{(k)}\Delta\tau P_{1,m,1}^{(k)}) \quad (37)$$

where $k=1,2,\ldots,K$; $m=1,2$,

$$P_{1,m,n}^{(k)}=\int_0^{\Delta\tau}\varphi_1(\tau)u^*(x,\xi,\tau)\,d\tau$$

$$=\frac{1}{\Delta\tau}\sqrt{\frac{Re}{4\pi}}\exp\left(-\frac{Re\bar{u}(x-\xi)}{2}\right)\int_0^{\Delta\tau}\sqrt{\tau}\exp\left(-\frac{Rer^2}{4\tau}-\frac{Re\bar{u}^2\tau}{4}\right)d\tau \quad (38)$$

$$P_{2,m,n}^{(k)}=\int_0^{\Delta\tau}\varphi_2(\tau)u^*(x,\xi,\tau)\,d\tau=P_{m,n}^{(k)}-P_{1,m,n}^{(k)} \quad (39)$$

$$Q_{1,m,n}^{(k)}=\int_0^{\Delta\tau}\varphi_1(\tau)q^*(x,\xi,\tau)\,d\tau=-\frac{Re(x-\xi)}{2\Delta\tau}P_{m,n}^{(k)}-\frac{\bar{u}Re}{2}P_{1,m,n}^{(k)} \quad (40)$$

$$Q_{2,m,n}^{(k)}=\int_0^{\Delta\tau}\varphi_2(\tau)q^*(x,\xi,\tau)\,d\tau=Q_{m,n}^{(k)}-Q_{1,m,n}^{(k)} \quad (41)$$

Analytical integration of (38) causes considerable complexity. Thus it is preferable to find the integral value employing Gauss quadrature, as the integral (38) contains no singularities.

Special attention is paid in this paper on the correct values of the calculations of the discrete BIE coefficients. If a non-negative number M is chosen to satisfy the conditions:

$$\exp(M)<\text{machine infinity}$$

$$\exp(-M)>\text{machine zero}$$

then it is possible to get a condition, limiting the time steps $\Delta\tau$ and the spatial steps $\Delta x=x_2^{(k)}-x_1^{(k)}$:

$$\Delta x\leqslant\Delta x_{\max}=\frac{4M}{Re} \quad (42)$$

$$\Delta\tau\leqslant\Delta\tau_{\max}=\frac{8}{Re\bar{u}_{\max}^2}\left(M+\sqrt{M^2-\frac{Re^2\bar{u}_{\max}^2\Delta x^2}{16}}\right) \quad (43)$$

Usually in calculations it was taken that $\Delta x\leqslant 0.1\Delta x_{\max}$, was the reason that the restriction of

the time step might be rewritten in the form:

$$\Delta\tau \leqslant \frac{16M}{Re\bar{u}^2_{\max}} \tag{44}$$

The number M is chosen depending on computer arithmetic and varies from 40 and higher. The conditions (42)–(44) are not connected with the Courant–Friedrichs–Levi condition of stability, but only reflect the coefficient calculations of correct conditions (without accuracy loss). Note that both the maximum time step and space step are inversely related to Reynolds number.

In BIE (37) the value $V_m^{(k)}$ is calculated according to (35), and the integral $S_m^{(k)}$ is already evaluated in accordance with linear interpolation with respect to τ and x for the velocities:

$$u^{(k)}(x,\tau) = \{\varphi_1(\tau)u_1^{(k)}\Delta\tau + \varphi_2(\tau)u_1^{(k)}\}\theta_1(x) + \{\varphi_1(\tau)u_2^{(k)}(\Delta\tau) + \varphi_2(\tau)u_2^{(k)}\}\theta_2(x) \tag{45}$$

$$u'^{(k)}(x,\tau) = \{\varphi_1(\tau)u_1'^{(k)}(\Delta\tau) + \varphi_2(\tau)u_1'^{(k)}\}\theta_1(x) + \{\varphi_1(\tau)u_2'^{(k)}(\Delta\tau) + \varphi_2(\tau)u_2'^{(k)}\}\theta_2(x) \tag{46}$$

While using the linear interpolating functions the convective velocity constant part may be found in the following way:

$$\bar{u}^{(k)} = 0.25(u_1^{(k)}(\Delta\tau) + u_1^{(k)} + u_2^{(k)}(\Delta\tau) + u_2^{(k)})$$

Having written the value of the velocity derivative from (45):

$$q = \frac{1}{x_2^{(k)} - x_1^{(k)}}[\varphi_1(\tau)\{u_2^{(k)}(\Delta\tau) - u_1^{(k)}(\Delta\tau)\} + \varphi_2(\tau)\{u_2^{(k)} - u_1^{(k)}\}]$$

the integral $S_m^{(k)}$ (36) is found numerically by employing the Gauss quadrature.

The algorithm of the Burgers' equations solution using the BEM technique may be represented in the following form:

(1) boundary and initial (at $t = 0$) conditions $u(x, 0)$, $q(x, 0)$ are specified;
(2) we carry out the step with respect to time $t := t + \Delta\tau$. Let us accept $w_n^{(k)} := u_n^{(k)}$,
  (a) calculate the convective velocity constant part $\bar{u}^{(k)}$ and then all the BIE coefficients;
  (b) find $V_m^{(k)}$ and $S_m^{(k)}$;
  (c) solve BIE (30) or (37) and find the values $u_n^{(k)}$, $q_n^{(k)}$;
  (d) check the root-mean-square residual with respect to velocity between two iterations:

$$RMS_u = \frac{1}{K}\sqrt{\sum_{k=1}^{K}(u_1^{(k)} - w_1^{(k)})^2}$$

  if $RMS_u \leqslant 10^{-9}$, then iterations are finished, in opposite case return to step (a);
  (e) find for the time t the velocity values $u(x, \tau)$ and their derivatives $q(x, \tau)$ in all internal Gauss nodal points;
  (f) compare numerical results with exact analytical ones and find the solution error.
(3) If $t \leqslant T$, then assign

$$u_n^{(k)}(\Delta\tau) := u_n^{(k)}; \qquad q_n^{(k)}(\Delta\tau) := q_n^{(k)}; \qquad n = 1, 2$$

and return to point (2). In the opposite case finish the calculations.

Here it should be noted that the global matrix $[A]$ formation procedure (see (15)) is completely analogous to what has been done already for the steady-state convective diffusion problem. On each iteration for the time t it is necessary to inverse matrix $[A]$, which has pentadiagonal structure as before. Note that iterations in point (2) are simple and numerical experiments show their high convergence rate.

In comparison with calculations of u, q in global nodal points, their calculation in inner Gauss node points is carried out in explicit form after having found the values in global nodes.

Although the coefficients of discrete BIE at each iteration should be calculated anew, nevertheless we should note that the total amount of non-zero coefficients is directly related to the number of global node points and not their quadrate as in the case of BEM. This circumstance is especially profitable in case of fine discretization when a lot of global nodes are employed.

RESULTS

According to the boundary element method with subdomains technique for solving linear steady-state convective–diffusion equation with variable Peclet number and non-linear unsteady Burgers' equations there have been composed programs for a personal computer. The achieved numerical results were compared with analytical solutions and this gave the possibility to judge the BEM accuracy. We carried out the investigation of the accuracy of the numerical solutions on the employment of refinement algorithm with respect to the variable convective velocity. Besides, the convergence steps with respect to space and time were studied.

Steady-state convection–diffusion

The analytical solution of linear one-dimensional equation of steady-state convection–diffusion with boundary conditions:

$$x = 0,\ u = u_0 \qquad x = 1,\ u = u_1$$

has the form:

$$u(x) = e^{Q(x)}\left[u_0 + \frac{(u_1 e^{-Q(1)} - u_0)\displaystyle\int_0^x e^{-Q(w)}\,\mathrm{d}w}{\displaystyle\int_0^1 e^{-Q(w)}\,\mathrm{d}w}\right] \tag{47}$$

where

$$Q(w) = \int_0^w P(x')\,\mathrm{d}x'$$

Correspondingly, the derivative $\mathrm{d}u/\mathrm{d}x$ may be written in the following way:

$$q(x) = P(x)u(x) + \frac{u_1 e^{-Q(1)} - u_0}{\displaystyle\int_0^1 e^{-Q(w)}\,\mathrm{d}w} \tag{48}$$

Depending on the character of the Peclet number change $P(x)$, the paper gives three different cases:

(a) Peclet number is constant $P(x) = P = \text{const}$. The steady-state convection–diffusion equation has a well-known solution:

$$u(x) = \frac{e^{Px} - 1}{e^P - 1} \tag{49}$$

The numerical solution with respect to the BEM technique is achieved with absolute computer accuracy because, on the one hand, all coefficients of the discrete BIE (12) are accurate, and on the other hand, the term $S_m^{(k)}$, considering the variable Peclet number contribution, is identically equal to zero.

(b) The Peclet number change goes linearly with respect to x:

$$P(x) = a_0 + a_1 x$$

In this case the Peclet number variable part also changes linearly over the every subdomain.

(c) The Peclet number change goes on with quadratic dependence:

$$P(x) = a_0 + a_1 x + a_2 x^2$$

Note that for the one-dimensional steady-state convective–diffusion problem only the error, connected with the $S_m^{(k)}$ term evaluation takes place, considering the Peclet number variable part contribution. In the case of two- and three-dimensional steady-state convective–diffusion problems additional errors appear at the discretization of Γ_k border of each subdomain Ω_k by boundary elements. The errors of this type are rather well investigated[1,2,7].

In *Table 1* we can see the comparison of analytical and numerical results while dividing the whole calculated domain into 10 parts of the same length. Numerical calculations were carried out both with consideration for Peclet number variable part (the data are marked as BEM-R) and without its consideration (BEM-N). Note that results achieved by using BEM-R are in good agreement with analytical values. The accuracy of the numerical results was estimated as maximum in modulus difference between numerical and analytical results:

$$\|u\| = \max|u_{\text{num}} - u_{\text{anal}}| \qquad \|q\| = \max|q_{\text{num}} - q_{\text{anal}}| \tag{50}$$

Such a norm is rather strict in comparison with the norm of the root-mean-square residual.

From *Table 1* it is seen that though the BEM-N gives rather high accuracy itself; nevertheless, the BEM-R increases considerably the numerical calculations accuracy. The employment of BEM-N and BEM-R schemes allows already on coarse mesh to achieve high accuracy for problems with non-monotonous variable change in the calculated domain.

Table 2 shows that practically in all the domain, with the exception of a small area at $x \to 1$, the values u and q are equal to zero. Note that in this case BEM-N models the exact solution significantly better, although the scheme BEM-R accuracy in comparison with the previous example has insignificantly decreased.

Table 1 Analytical and numerical results for one-dimensional steady-state convective diffusion problem: $P = -50x^2$, $u(0) = 0$, $u(1) = 1$

		u			q		
x	$P(x)$	Analytical	BEM-N	BEM-R	Analytical	BEM-N	BEM-R
0.000	0.00	0.000	0.000	0.000	47.85	45.04	47.86
0.100	−0.50	4.726	4.467	4.727	45.49	42.81	45.50
0.200	−2.000	8.666	8.226	8.669	.30.52	28.59	30.53
0.300	−4.50	10.331	9.854	10.335	1.36	0.70	1.36
0.400	−8.00	9.046	8.680	9.049	−24.51	−24.40	−24.53
0.500	−12.50	6.168	5.968	6.170	−29.25	−29.56	−29.25
0.600	−18.00	3.710	3.628	3.709	−18.93	−20.26	−18.90
0.700	−24.50	2.337	2.308	2.336	−9.41	−11.52	−9.37
0.800	−32.00	1.651	1.641	1.650	−4.98	−7.49	−4.95
0.900	−40.50	1.258	1.256	1.258	−3.11	−5.83	−3.09
1.000	−50.00	1.000	1.000	1.000	−2.15	−4.96	−2.14

	BEM-N	BEM-R
$\lg\|u\|$	−0.3208	−2.466
$\lg\|q\|$	0.4495	−1.344

Table 2 Analytical and numerical results for one-dimensional steady-state convective diffusion problem: $P = 50x^2$, $u(0) = 0$, $u(1) = 1$

		u			q		
x	$P(x)$	Analytical	BEM-N	BEM-R	Analytical	BEM-N	BEM-R
0.000	0.00	0.000	0.000	0.000	0.00	0.00	0.00
0.100	0.50	0.000	0.000	0.000	0.00	0.00	0.00
0.200	2.00	0.000	0.000	0.000	0.00	0.00	0.00
0.300	4.50	0.000	0.000	0.000	0.00	0.00	0.00
0.400	8.00	0.000	0.000	0.000	0.00	0.00	0.00
0.500	12.50	0.000	0.000	0.000	0.00	0.00	0.00
0.600	18.00	0.000	0.000	0.000	0.00	0.00	0.00
0.700	24.50	0.000	0.000	0.000	0.00	0.00	0.00
0.800	32.00	0.000	0.000	0.001	0.01	0.01	0.02
0.900	40.50	0.011	0.011	0.017	0.44	0.44	0.68
1.000	50.00	1.000	1.000	1.000	50.00	50.00	50.00

	BEM-N	BEM-R
$\lg\|\|u\|\|$	−9.477	−2.240
$\lg\|\|q\|\|$	−8.020	−0.632

In cases with sharply changing values and employing linear interpolation (16) for calculating $S_m^{(k)}$, (18) is not completely exact, it is likely to use more exact, but also and more difficult interpolating functions, and this would by all means lead to sharp complication of the algorithm itself. In the last example, where almost everywhere $u = 0$ and $q = 0$, we may note that the value $S_m^{(k)}$ contribution from (13) must tend to zero and not be artificially raised as in case of BEM-R with employment of linear interpolation. Thus the BEM-N scheme, where $S_m^{(k)} = 0$ gives much higher accuracy, than BEM-R. Besides, with the Peclet number growth, the accuracy of BEM-N will be still higher.

In *Tables 3* and *4* we can see the comparison of numerical and analytical results also at quadratic changes of the Peclet number. We should again mention the higher results accuracy, found by use of the BEM-R in comparison with BEM-N. This is particularly seen in *Table 3*. Though the BEM-R technique almost in all nodal points gives much better than BEM-N technique convergence with analytical results; nevertheless, close to the sharp change values domain, there takes place a little disagreement, which leads to practically the same accuracy of both methods while employing norms (50). It is clear that the usage of norm with respect to the root-mean-square residual would have shown a higher accuracy of BEM-R method.

The high degree of numerical and analytical results convergence made it necessary to show them in the form of tables and not in graphical form as is done in *Figure 1*. Here we see the comparison of numerical and analytical results for $P(x) = -50x$, $P(x) = -50$, $P(x) = 50x$ and also at boundary conditions $u(0) = 0$, $u(1) = 1$. Despite the complicated both $u(x)$ and $q(x)$ change character, the lines (analytical calculation) and points (numerical calculation) on *Figure 1* practically merge and it is impossible to define which of the numerical methods is more exact.

The next Figures show the results of the BEM-N and BEM-R methods numerical accuracy investigation depending on the Peclet number change character and the amount of the investigated nodes. The data shown in *Figure 2* demonstrate the accuracy of the numerical methods depending on coefficient a_1 at linear Peclet number change $P(x) = a_1 x$. At $a_1 \to 0$ the numerical calculations errors tend to zero, as the BEM method in case of $P = 0$ becomes accurate. With the growth of $|a_1|$ the errors increase and it is possible to conclude that the errors for BEM-N grow quicker than for BEM-R. In the investigated Peclet number range the BEM-R

Table 3 Analytical and numerical results for one-dimensional steady-state convective diffusion problem: $P = -50x^2$, $u(0) = 1$, $u(1) = 1$

		u			q		
x	$P(x)$	Analytical	BEM-N	BEM-R	Analytical	BEM-N	BEM-R
0.000	0.00	1.000	1.000	1.000	47.85	45.04	47.86
0.100	−0.50	5.709	5.450	5.711	45.00	42.31	45.01
0.200	−2.00	9.541	9.101	9.544	28.77	26.84	28.78
0.300	−4.50	10.969	10.491	10.972	−1.51	−2.17	−1.51
0.400	−8.00	9.390	9.024	9.393	−27.27	−27.15	−27.28
0.500	−12.50	6.293	6.092	6.294	−30.80	−31.11	−30.81
0.600	−18.00	3.737	3.655	3.736	−19.42	−20.75	−19.39
0.700	−24.50	2.341	2.312	2.339	−9.50	−11.60	−9.45
0.800	−32.00	1.651	1.642	1.651	−4.98	−7.49	−4.95
0.900	−40.50	1.258	1.256	1.258	−3.11	−5.83	−3.09
1.000	−50.00	1.000	1.000	1.000	−2.15	−4.96	−2.14

	BEM-N	BEM-R
$\lg\|\|u\|\|$	−0.3208	−2.474
$\lg\|\|q\|\|$	+0.4495	−1.325

Table 4 Analytical and numerical results for one-dimensional steady-state convective diffusion problem: $P = 50x^2$, $u(0) = 1$, $u(1) = 1$

		u			q		
x	$P(x)$	Analytical	BEM-N	BEM-R	Analytical	BEM-N	BEM-R
0.000	0.00	1.000	1.000	1.000	−2.86	−2.91	−2.86
0.100	0.50	0.727	0.724	0.727	−2.50	−2.55	−2.50
0.200	2.00	0.510	0.505	0.510	−1.84	−1.90	−1.84
0.300	4.50	0.356	0.351	0.356	−1.26	−1.33	−1.26
0.400	8.00	0.253	0.247	0.253	−0.84	−0.93	−0.84
0.500	12.50	0.184	0.179	0.184	−0.56	−0.67	−0.56
0.600	18.00	0.138	0.133	0.138	−0.38	−0.51	−0.38
0.700	24.50	0.106	0.102	0.106	−0.26	−0.41	−0.26
0.800	32.00	0.084	0.080	0.084	−0.18	−0.34	−0.17
0.900	40.50	0.078	0.075	0.083	0.28	0.11	0.50
1.000	50.00	1.000	1.000	1.000	47.14	47.09	47.14

	BEM-N	BEM-R
$\lg\|\|u\|\|$	−2.238	−2.264
$\lg\|\|q\|\|$	−0.781	−0.656

method gives the accuracy $1.5 \cdots 2$ orders higher than the method of BEM-N. This refers to the value u and q as well. The exception is made only by the diapason $a_1 > 0$ for boundary conditions $u(0) = 0$, $u(1) = 1$ (see *Figure 2b*), where BEM-N accuracy sharply increases and grows with the growth of a_1 but does not decrease as in the BEM-R method. The reason of this phenomenon has been discussed above.

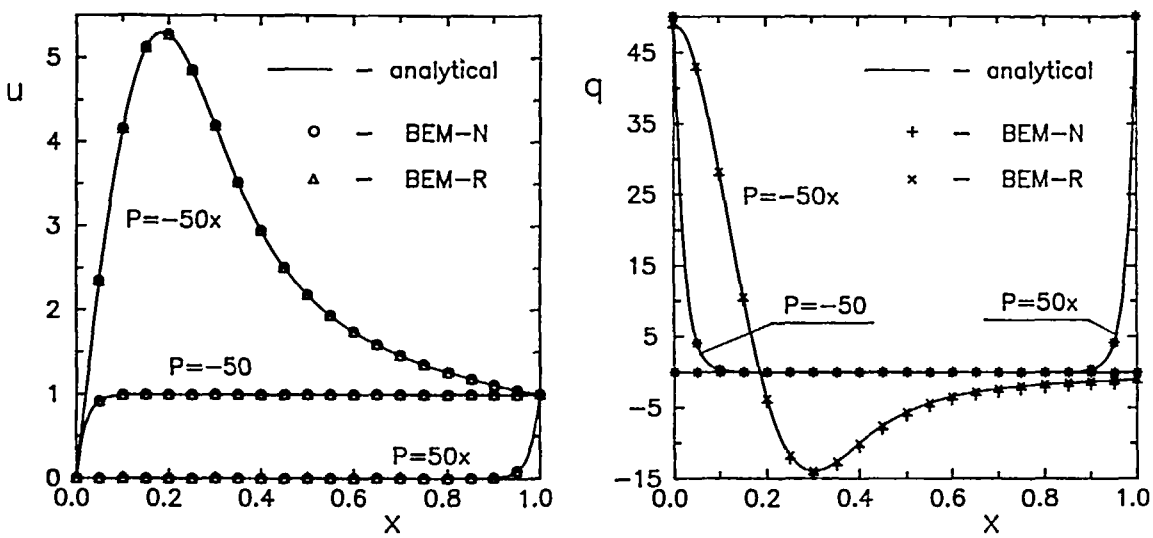

Figure 1 Comparison of analytical and numerical results for one-dimensional convective diffusion problem

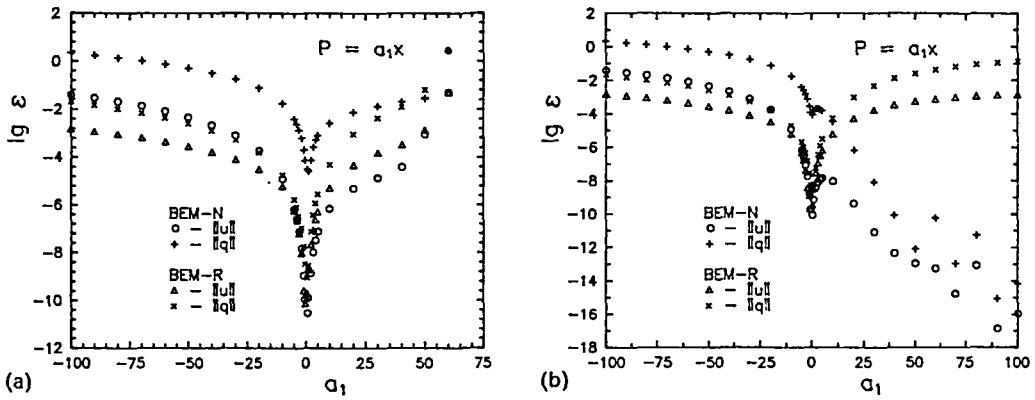

Figure 2 Numerical errors for linear change of Peclet number: (a) $u(0) = 0.5$, $u(1) = 1$; (b) $u(0) = 0$, $u(1) = 1$

Figure 3 shows similar data for the quadratic dependency of the Peclet number $P = a_2x^2$ at different boundary conditions. Attention is drawn to the similarity of the achieved results both for linear and quadratic dependency of Peclet number. Nevertheless, for stronger non-linearity (quadratic dependency of P) the difference between the accuracy of BEM-R and BEM-N methods becomes higher, than at weaker (linear dependency). Thus, the advantages of the BEM-R method become more vivid for the problems with more significant non-linearities.

Figure 4 shows the influence of the spatial co-ordinate step $h = \Delta x$ on the accuracy of the numerical methods. Numerical results may be approximated rather accurately by the dependence of the type:

$$\varepsilon = ch^m \tag{51}$$

The degree index in (51) reflects the numerical algorithm convergence order. We call attention to the fact that the BEM-R scheme convergence approaches the fourth order with respect to h

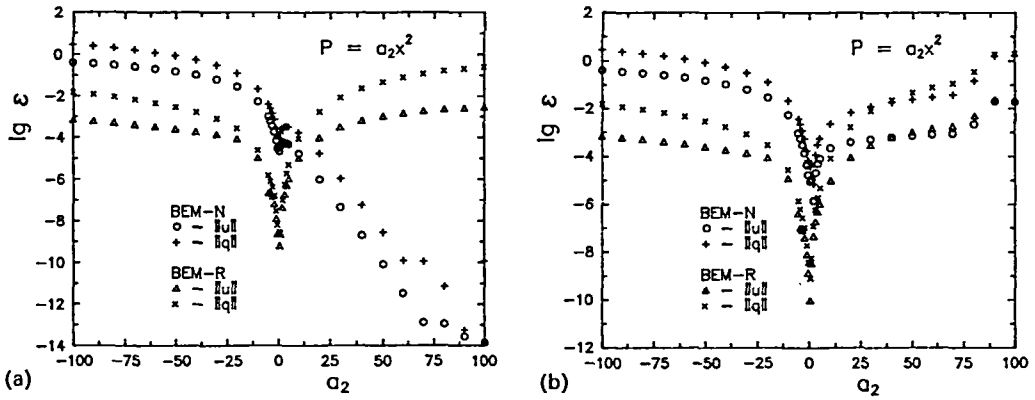

Figure 3 Numerical errors for quadratic change of Peclet number: (a) $u(0) = 0$, $u(1) = 1$; (b) $u(0) = 0.5$, $u(1) = 1$

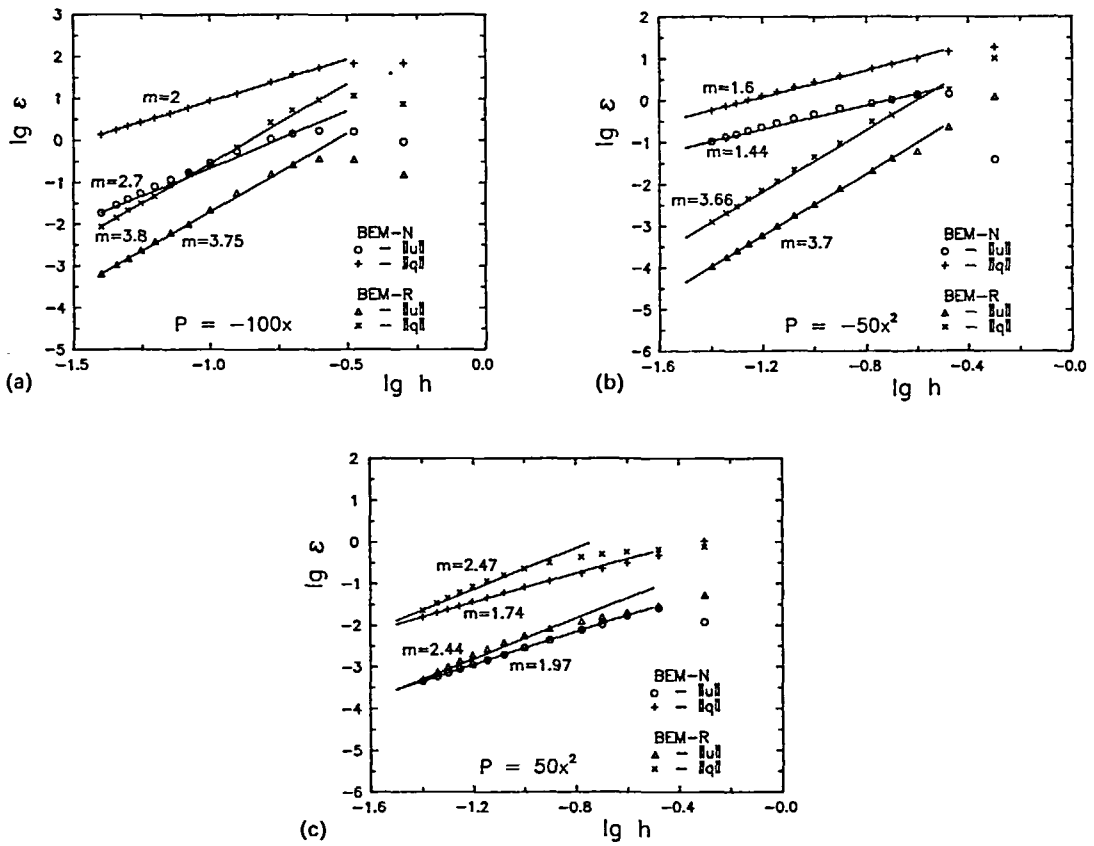

Figure 4 BEM-N and BEM-R errors versus space step: (a) $u(0) = 0$, $u(1) = 1$; (b) $u(0) = 0.5$, $u(1) = 1$; (c) $u(0) = 0.5$, $u(1) = 1$

and the BEM-N scheme convergence does not exceed the third accuracy order. At very coarse domain discretization ($K \leqslant 4$) the numerical scheme convergence with respect to h stops to be described by the (51) type dependence, and the errors become lower. But again the effect of the BEM-R scheme employment for the problems with the stronger non-linearity (see *Figure 4b*), where the results with the quadratic dependency of the Peclet number are represented, becomes more noticeable. In this case the degree index for BEM-R constitutes 3.7 (for $\|u\|$), that is significantly higher than the analogous value $m = 1.44$ for BEM-N. Nevertheless, for the problems with a large positive Peclet number $P = 50x^2$ (*Figure 4c*), the advantages of BEM-R over BEM-N are equal to zero only because of the fact that the error norm (50) is used in this paper.

It is well known that while solving non-linear steady-state convective diffusion problems by finite difference and finite element methods there appear numerical stability problems. In many cases there exist strong limitations on cell Peclet number $P_c \leqslant 2$ and if it becomes more than 2, then the numerical solution oscillates. In this paper calculations have been carried out at far higher Peclet numbers, but no numerical oscillations were found. This undeniably gives advantages to the proposed method in comparison with other methods. On the other hand, the distinctive feature of this method (both BEM-N and BEM-R) is its high accuracy at relatively coarse discretization of the domain.

Although the aim of this investigation did not include the comparison of the accuracy of the given method with that of the finite difference and finite element methods, nevertheless it should be noted that the BEM method is highly competitive with other numerical methods.

Burgers' equation

If the investigation of the steady-state convective diffusion equations solutions discloses the BEM technique convergence with respect to the spatial coordinate, then the analysis of the one-dimensional Burgers' equation gives us the opportunity to find out the approximation accuracy of the numerical scheme with respect to time. The analytical solution of one-dimensional Burgers' equation with boundary:

$$\lim_{x \to -\infty} u(x,t) = 1, \qquad \lim_{x \to +\infty} u(x,t) = 0 \tag{52}$$

and initial conditions:

$$\begin{cases} u_0(x) = u(x,0) = 1 & \text{at } x \leqslant 0 \\ u_0(x) = u(x,0) = 0 & \text{at } x > 0 \end{cases} \tag{53}$$

has the form[10]:

$$u(x,t) = \frac{\operatorname{erfc}\left[(x-t)\sqrt{\frac{Re}{4t}}\right]}{\operatorname{erfc}\left[(x-t)\sqrt{\frac{Re}{4t}}\right] + \exp\left[\frac{Re}{2}\left(x-\frac{t}{2}\right)\right]\operatorname{erfc}\left(-x\sqrt{\frac{Re}{4t}}\right)} \tag{54}$$

where $\operatorname{erfc}(\cdot)$ are error functions. Correspondingly, the derivative with respect to the spatial coordinate x may be written in the following way:

$$q(x,t) = -\frac{\exp\left[\frac{Re}{2}\left(x-\frac{t}{2}\right)\right]\left[\sqrt{\frac{Re}{\pi t}}\exp\left(\frac{x^2 Re}{4t}\right) + \frac{Re}{2}u(x,t)\operatorname{erfc}\left(-x\sqrt{\frac{Re}{4t}}\right)\right]}{\operatorname{erfc}\left[(x-t)\sqrt{\frac{Re}{4t}}\right] + \exp\left[\frac{Re}{2}\left(x-\frac{t}{2}\right)\right]\operatorname{erfc}\left(-x\sqrt{\frac{Re}{4t}}\right)}$$

In numerical calculations the domain should be confined as $x_{min} \leqslant x \leqslant x_{max}$. The lower and the higher limits are chosen to be able to satisfy the boundary condition (52) for all time values $0 \leqslant t \leqslant T$, where T is the time at which all the calculations are stopped.

The Burgers' equation numerical modelling at boundary (52) and initial (53) conditions is interesting for the fact that because of its non-linearity the shock wave is formed and it propagates along x in the course of time. At high Reynolds numbers the shock wave front turns out to be very steep and thus in various numerical methods there appear some difficulties in modelling this front. Usually this phenomenon is accompanied by strong numerical oscillations before and after passing the shock wave front which sometimes may lead to the numerical scheme instability. Besides, in numerical methods, employing upwind scheme, there appears the problem of artificial viscosity which leads to diffusion of the shock wave steep front. Very often in computational fluid dynamics, the numerical algorithm efficiency is judged by the capability of the method to simulate the shock wave propagation, which is governed by Burgers' equation with boundary (52) and initial (53) conditions. In the present problem almost all the difficulties in Navier–Stokes equations modelling, except pressure evaluation and the presence of negative velocity zones, are available. But we shall not consider the pressure evaluation problem in this paper. The problem of numerical solution stability preservation at velocities $u < 0$ will be considered below.

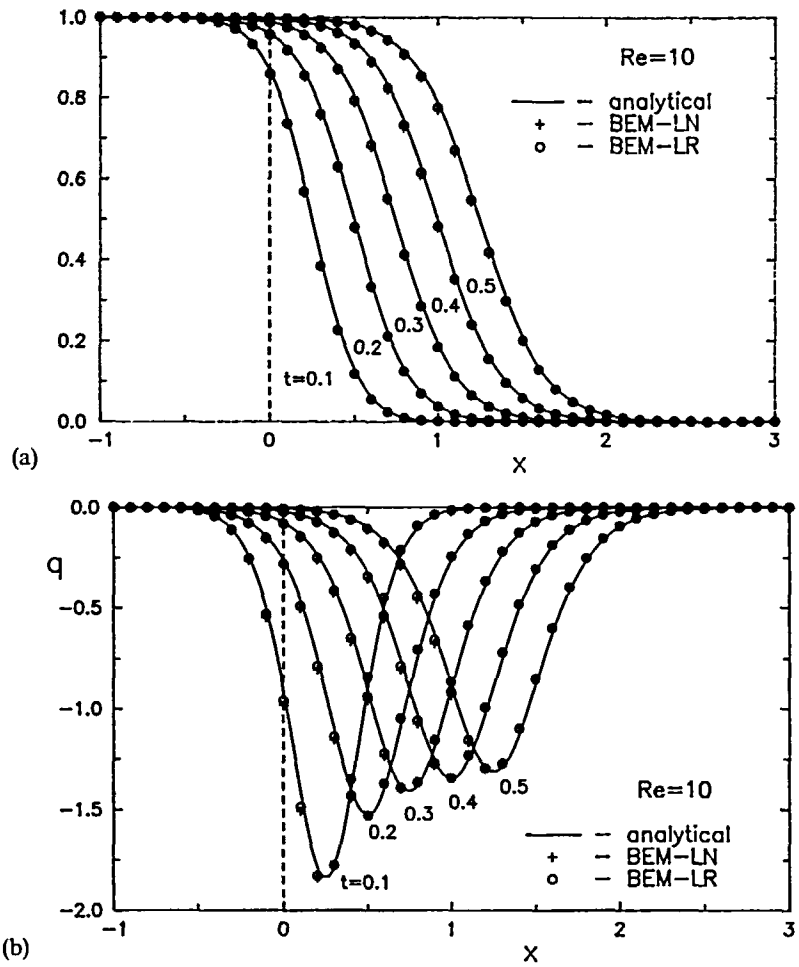

Figure 5 Comparison of analytical and numerical solutions of Burgers' equation: $Re = 10$, $\Delta x = 0.1$, $\Delta\tau = 0.05$

In numerical modelling of one-dimensional Burgers' equation an integration with respect to time is often carried out by the Runge–Kutta scheme to reduce to a minimum the solution with respect to time errors. In this case only the spatial approximation accuracy of the numerical scheme is investigated. In this paper aiming, at an optimum BEM method approach for solving applied (unsteady two- and three-dimensional) problems, the integration with respect to time was not carried out by the Runge–Kutta scheme, but by employing the fundamental solution depending on time. It is absolutely clear that employment of the Runge–Kutta scheme would have given much more accurate results in this paper, but in two and three-dimensional cases they lead to unjustified expenditure of computer resources. It is preferable to use either the formulation described here or any other formulation with the use of fundamental solutions for steady-state equations, e.g. the DRBEM method[11] or finite-differential approximation of the second order with respect to time for unsteady term.

In the present investigation numerical calculations have been carried out at Reynolds numbers up to 10,000 (at $Re > 1000$ only the BEM scheme numerical stability has been investigated). *Figures 5*, *6* and *7* show the comparison of analytical and numerical Burgers' equation solutions at Reynolds numbers 10, 100 and 1000 respectively. Numerical results have been achieved for

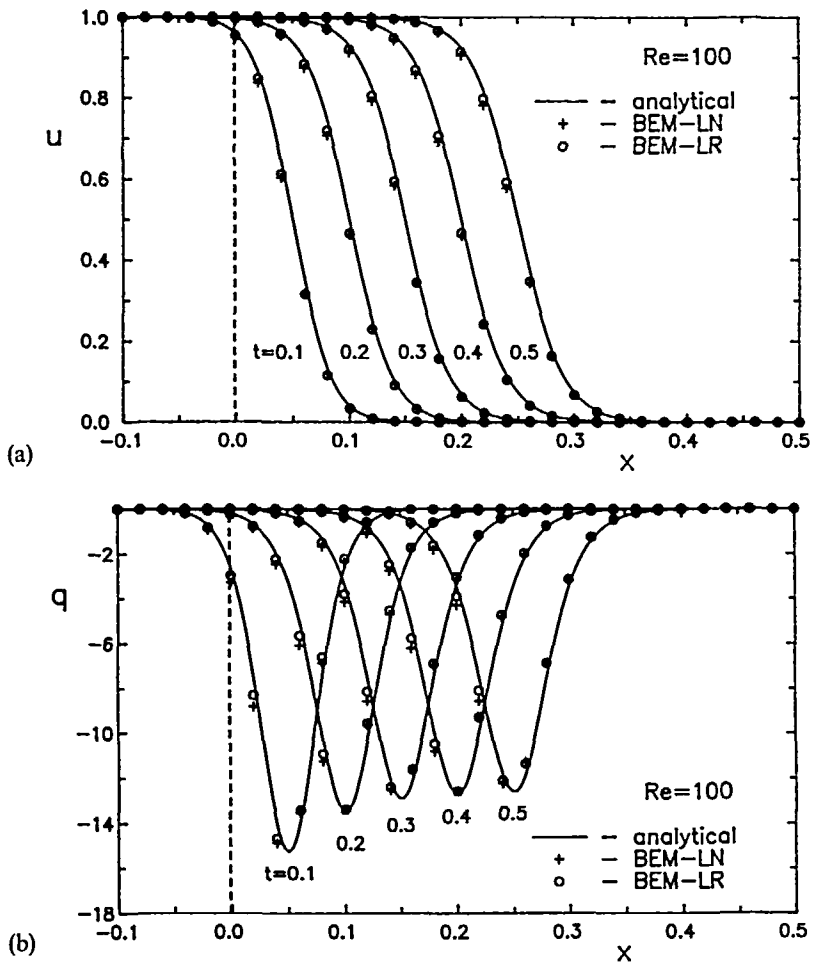

Figure 6 Comparison of analytical and numerical solutions of Burgers' equation: $Re = 100$, $\Delta x = 0.02$, $\Delta \tau = 0.01$

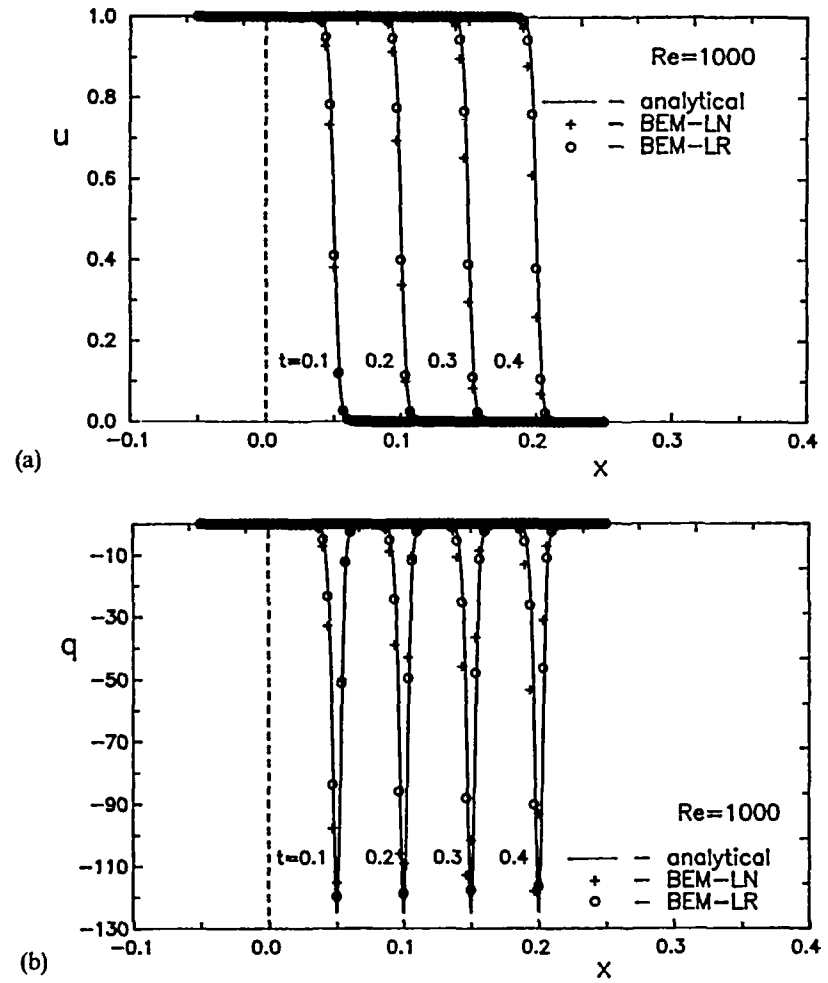

Figure 7 Comparison of analytical and numerical solutions of Burgers' equation: $Re = 1000, \Delta x = 1/300, \Delta\tau = 0.0025$

linear interpolation with respect to time both with consideration for variable convective velocity (the data are marked as BEM-LR) and without its consideration (BEM-LN). If for the low Reynolds number $Re = 10$ the time step $\Delta\tau = 0.05$ has been employed, then for $Re = 100$ the step constituted $\Delta\tau = 0.01$, and for $Re = 1000$ it was already $\Delta\tau = 0.0025$. Although using large time steps the numerical stability has not been destroyed, the shock wave front for numerical solutions turned out to be removed referring to the accurate solution. The greater the time step, the more removal there happened to be. It was especially noticeable at high Reynolds numbers. It is typical that to get a satisfactory numerical solution with respect to time (so that the numerical error did not increase with respect to time), it was necessary to realize Courant–Friedrichs–Levi condition[12]:

$$CFL = \frac{u_{max}\Delta\tau}{\Delta x} \leqslant 1 \tag{55}$$

Although the other methods of finite differences and finite elements possess numerical stability, they showed more or less acceptable accuracy only when the condition (55) was fulfilled.

Condition (55), at which it is possible to achieve solutions of high accuracy with respect to time, is not the BEM scheme stability condition. If it is necessary to obtain only accurate steady-state solutions the numerical calculations may be carried out with much larger time steps, than from (55).

For all three Reynolds numbers $Re = 10$, 100 and 1000 the numerical results demonstrate a high degree of convergence with analytical solutions for all investigated time intervals (*Figures 5, 6* and *7*). The most curious is the good convergence of the results for high Reynolds number $Re = 1000$, where a very steep shock wave front is formed (*Figure 7*), whereas not more than four node points are on the front itself. In comparison with the majority of other numerical methods BEM does not lead to numerical oscillations before and after shock wave passing. It is also necessary to mark good agreement of numerical and analytical values of the velocity derivative $q = \partial u/\partial x$. In *Figures 5, 6* and *7* the numerical results (points) practically coincide with the analytical ones (curves), and so it is difficult to decide which method is more accurate: BEM-LN or BEM-LR? Data shown in *Figures 8a* and *8b* help to answer this question to some extent. For Reynolds numbers $Re = 10$ and 100, numerical solutions errors decrease in time for

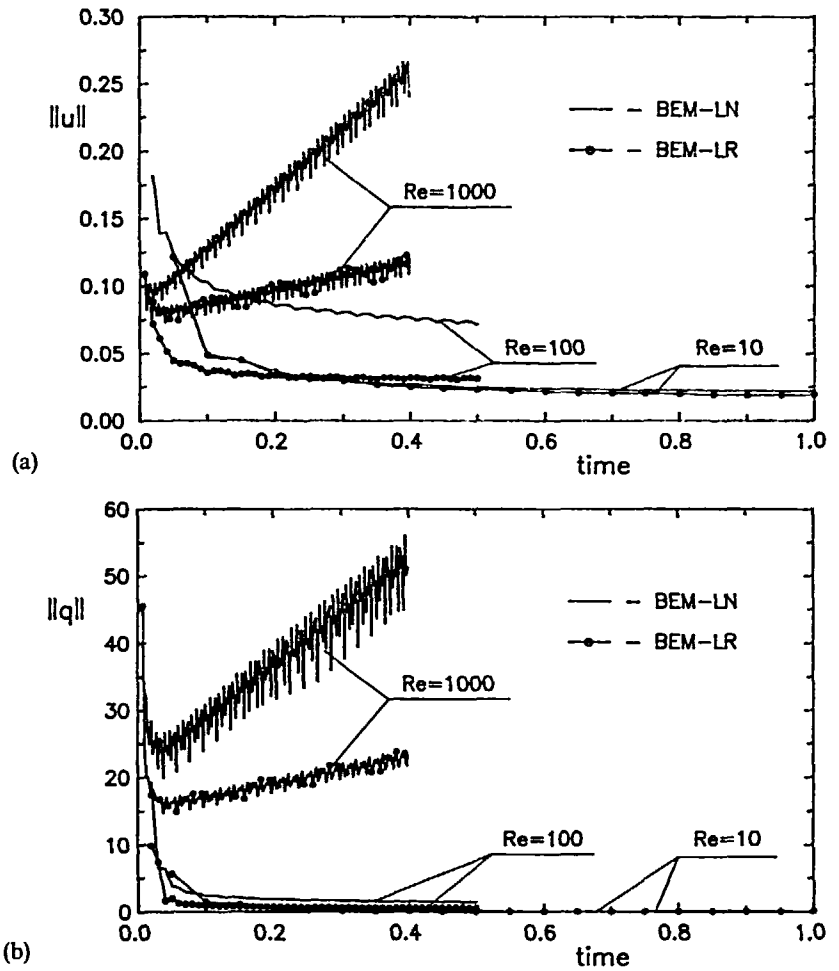

Figure 8 Time evolution of BEM-LN and BEM-LR numerical errors

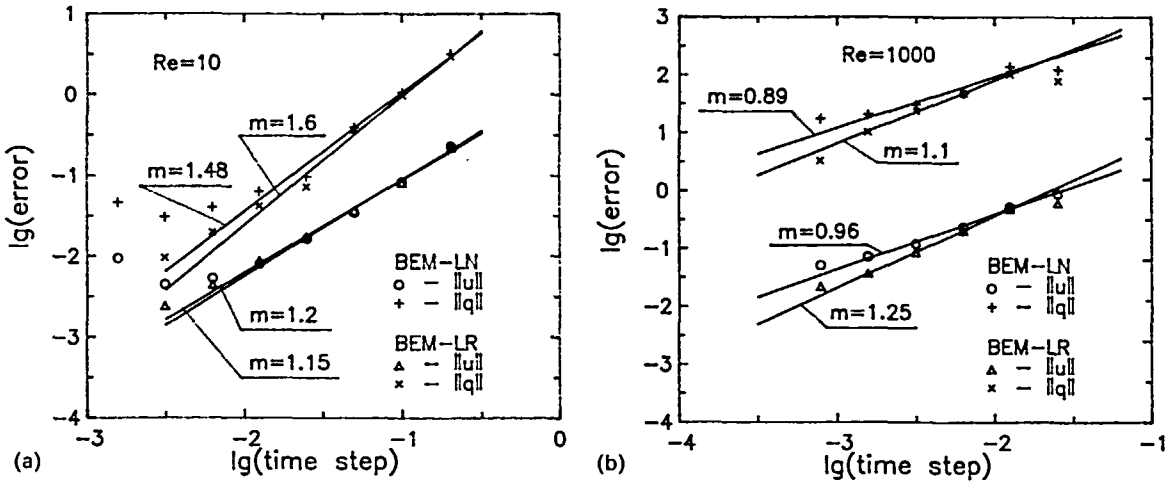

Figure 9 BEM-LN and BEM-LR numerical errors versus time step: (a) $Re = 10$, $\Delta x = 0.1$, $T = 0.2$; (b) $Re = 1000$, $\Delta x = 0.005$, $T = 0.025$

both BEM-LN and BEM-LR schemes. Nevertheless, for $Re = 1000$, numerical errors with respect to time do not decrease; moreover, the BEM-LN method error grows twice as fast as the BEM-LR error (see *Figure 8*). It is clear that the BEM-LR method efficiency is more noticeable for high numbers, $Re \geqslant 100$, where decrease of the numerical errors more than twice is observable at employed time steps.

Figures 9a and *9b* show the results of the investigation of the numerical calculations errors for BEM-LN and BEM-LR depending on time step at $t = 0.2$ for $Re = 10$ and $t = 0.025$ for $Re = 1000$. Up to some critical time step, the numerical calculations error decreases in degree dependence:

$$\text{error} = C\Delta\tau^m \tag{56}$$

and then the step $\Delta\tau$ decrease does not already lead to the error decrease. The reason of such phenomenon lies in the fact that at reaching a critically small time step errors begin to dominate connected with too small a spatial step. Just by employing $\Delta\tau$ steps, less than critical time step, it is possible to disclose the BEM-LR scheme advantages over BEM-LN to the full. But still this possibility is significantly limited by the fact that the critical time step itself depends on the spatial step h.

It is necessary to note that schemes BEM-LN and BEM-LR are the schemes of approximately the first order of accuracy with respect to time (*Figure 9*). The same first order accuracy with respect to time curve BEM-CN and BEM-CR schemes, which employ constant interpolation with respect to time functions (*Figure 10*). There is nothing surprising in the fact that schemes employing constant and linear interpolating functions, have approximately the same order of accuracy with respect to time. The reason lies in strong non-linear character of velocity change depending on time while passing the shock wave front. At some time intervals constant interpolation becomes desirable and at some—by linear functions. Thus, on the whole, the accuracy of numerical schemes with linear interpolating functions reduces to the first order. Still we should wait, that at more favourable boundary and initial conditions linear time schemes will give higher convergence degree, than constant schemes with respect to time interpolating functions.

It has already been mentioned above that the employment efficiency of the algorithm of refinement with respect to variable convective velocity may be demonstrated only at time step $\Delta\tau$, less than critical time step. So *Figure 11* represents the results of numerical investigation of

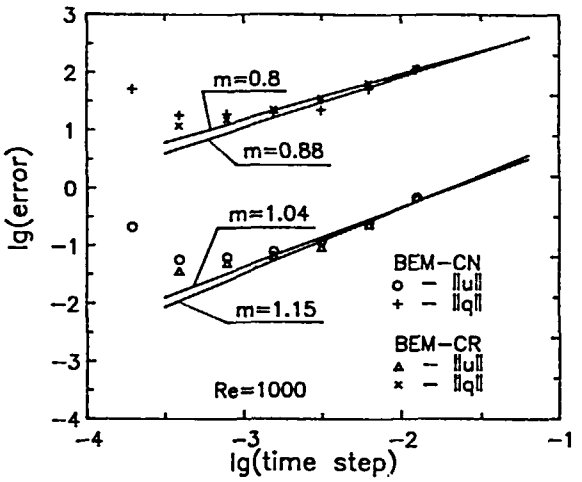

Figure 10 BEM-CN and BEM-CR numerical errors versus time step: $Re = 1000$, $\Delta x = 0.005$, $T = 0.0125$

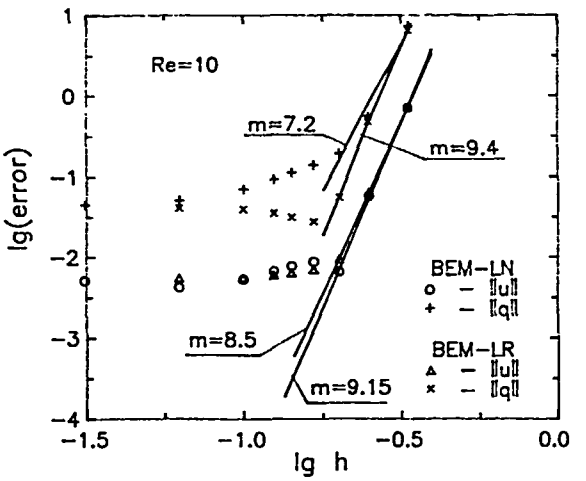

Figure 11 BEM-LN and BEM-LR numerical errors versus space step: $Re = 10$, $\Delta\tau = 0.005$, $T = 0.1$

spatial accuracy of numerical BEM-LN and BEM-LR schemes for time step $\Delta\tau = 0.005$, which is less than critical time step. The fact that in a rather narrow zone of spatial step we observe high convergence degree (*Figure 11*) has engaged our attention. It is partially explained by employing the norms (50). For the nodal points, situated already in some distance from the shock wave front and constituting the overwhelming majority of the employed nodes, the BEM scheme gives the ideal results accuracy. The exception is made only by those nodes, which are on the shock wave front. But the maximum error for velocity from (50) may constitute only 1, which is observed at very coarse discretization. So a very sharp error change is observed in a rather narrow interval along h. In case the norm with respect to root-mean-square residual were employed, the error change character would not have been so steep.

At reaching the certain lower critical step Δx the numerical solutions error stops reducing and at spatial step decreasing stays approximately at the same level (*Figure 11*). This error level corresponds to the discretization with respect to time error, and may be decreased only at time step reduction. So it is possible to say that by using the above BEM schemes for solving transient problems it is necessary to stick to approximately certain correlations between the spatial and time steps, which give almost the same contribution to the joint total error. Otherwise the non-observance of this correlation would not lead to the reduction of the numerical calculations errors, no matter how fine the step either with respect to x or t would be used.

In conclusion we should mention that though at each time step it is necessary to solve BIE (30) or (37) at iteration way, nevertheless, numerical investigation shows very high iteration method convergence degree for BEM (*Figure 12*). As a rule, the amount of necessary iterations for reducing the error up to 10^{-9} did not precede 15 at the employment of rather large time steps for $Re = 1000$. *Figure 12* shows that approximately the same convergence character occurs both at low ($Re = 10$) and high Reynolds numbers ($Re = 1000$). Besides, the employment of the algorithm of considering the contribution of the variable convective velocity does not lead to perceptible increase of the amount of iteration convergence but the time of the program fulfilment on the base of BEM-LR is about 1.5 times higher than in the analogous program on the base of BEM-LN because of the necessity to calculate $S_m^{(k)}$ at each iteration.

The above example of the shock wave propagation does not cause the appearance of negative velocity values, and so it is difficult to answer the question about the BEM algorithm numerical

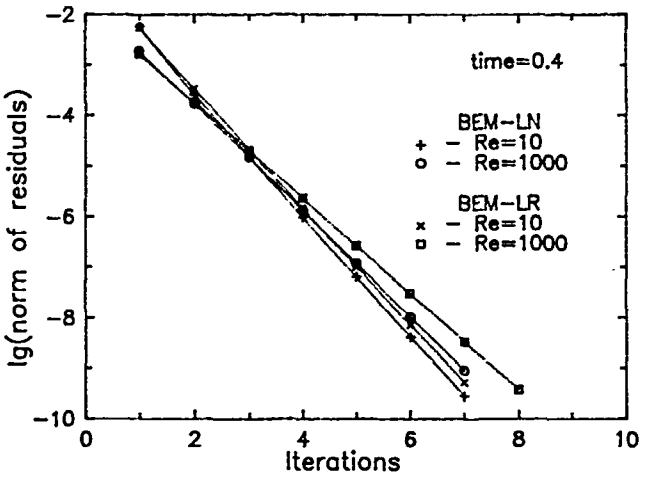

Figure 12 Iteration process convergence at time $t = 0.4$

stability, having analysed only this example. The aim in this paper has been to carry out additional numerical calculations of Burgers' equation at boundary and initial conditions, leading to the appearance of negative velocity zones, for example, to accept boundary conditions in the form:

$$\lim_{x \to -\infty} u(x, t) = 1 \qquad \lim_{x \to +\infty} u(x, t) = -1$$

and the initial:

$$\begin{cases} u_0(x) = u(x, 0) = 1 & \text{at } x \leqslant 0 \\ u_0(x) = u(x, 0) = -1 & \text{at } x > 0 \end{cases}$$

The task of such numerical calculations came only to checking up the BEM scheme numerical stability at positive and negative velocity values alternation. Calculations for Reynolds numbers up to 10,000 showed the BEM scheme stability which is certainly a great advantage of this method. Thus the necessity of employing additional methods for increasing the numerical method stability disappears; for example, there is no need to use the upwind scheme.

All these data give hope that the BEM method would be successfully employed for more complicated unsteady two- and three-dimensional fluid dynamics, heat and mass transfer problems and already in the near future there would appear computer programs, which would not give way to the methods of finite differences, finite elements and spectral methods.

CONCLUSION

The paper describes the new BEM technique for solving equations of the convection–diffusion type. The whole domain is divided into K subdomains, in which governing equations have been linearized by representing the convective velocity in the form of the constant and variable velocity parts sum. For the linearized differential operator either the known fundamental solution of convection–diffusion equation is used or, in the more general case at studying the Navier–Stokes equations, it is necessary to find special fundamental solutions.

The employment of such fundamental solutions in a steady-state case allows us to obtain a boundary integral equation, involving the variables only on the border of the subdomain if the variable convective velocity contribution is not taken into consideration. In a more complicated

case, when the variable convective velocity is taken into account, BIE is more accurate. Employing this method to unsteady problems leads to the appearance of an integral over the subdomain, including that known from the previous time step variables. Written for every subdomain the BIE in two or three-dimensional cases may be discretized similarly as it is customary in the BEM technique. Then the formation of the global matrix takes place and it turns out to be of block band structure. In general cases both the discretized and global equations system for each subdomain BIE are non-linear. The employment of the Newton–Raphson method for solving non-linear system of equations is obviously less efficient than the employment of a simple iterative procedure, described in the paper. The reason lies in the complex character of discretized BIE coefficients dependence on the averaged convective velocity. Thus it is very difficult to define Jacobian matrix elements for the Newton–Raphson method. Besides, the employment of the iterative procedure gives a very high convergence rate.

The evaluation of the convective term, as in the BEM method, gives, most obviously, stability to this method and does not lead to loss of accuracy. On the contrary, there are opportunities of increasing the accuracy method. Carrying out the analysis of BEM scheme numerical stability for non-linear problems represents a very complicated task. So the check-up of the scheme stability at the alternation of the zones with positive and negative velocity values has been carried out numerically. This check showed the scheme to be stable up to Reynolds numbers $Re = 10{,}000$. This gives hope, that BEM method might be developed for calculating turbulent flows as well.

The presented BEM method was employed for numerical solution of one-dimensional steady-state convection–diffusion and Burgers' equations. The numerical solution results are compared with analytical solutions of these problems, this making it possible to find out the numerical methods error. It is shown that the half reduction to fragments Δx for steady-state convection–diffusion equation leads to the error reduction from 4 to 14 times. The highest accuracy is achieved with the algorithm considering the Peclet number variable contribution. It is noted that the only error at solving one-dimensional steady-state convection–diffusion equation appears by taking this part into account or not. By utilizing the BEM method for two and three-dimensional steady-state problems there would appear additional errors of discretization of the subdomain border by boundary elements but they might be lessened by employing high order boundary elements.

For one-dimensional Burgers' equations the problem of shock wave propagation with known analytical solution is analysed. The BEM method high accuracy is shown even for the high Reynolds number $Re = 1000$, where a shock wave with a very steep front is formed. In comparison with many other numerical methods, the proposed method does not lead to oscillations immediately before and after the shock wave front passing and does not give any perceptible artificial diffusion, leading to diffuse the shock wave front. The ability to find numerical solutions with high accuracy in the domain of very large gradients is very useful at numerical modelling of flows with thin boundary layers, emerging at high Reynolds numbers.

All the above data gives hope that in the near future on the basis of this numerical method there would be created highly efficient methods of numerical modelling of laminar and turbulent flows with high Reynolds numbers, if only an efficient enough method of evaluating pressure is employed.

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